During his sojourn at Tonnerre M. Denizet saw Madame Quingeri and Madame Leprince, two frequent visitors there, saying in a lofty way that he never forgot his old friends.

The day of his departure he finished up the business and asked me to dinner. He was very affable and, I must say in justice to him, he paid me special attention. I do not know whether, with patience, I shall ever see the end of this business. If I do, I shall need to have shown a lot of it.

I now come to the second part of your letter.

I have always blamed the misplaced pride of Bailly[1] and Condorcet.[2] As to Duséjour[3] I was under the impression that he had been executed, but you tell me that he died of fear of terrorism.[4]

. . . I was right about complaining about Madame de Marcheval and Madame de Lesseville.[5] These two females[6] have never ceased to stir up my wife against me. I made a short call at Auteuil in August 1779[7] and took supper at my aunt's, but I did not succeed in making any better impression. She had already got fixed in my wife's head the necessity for leaving me and managed to obtain the *lettre de cachet*,[8] on the 30th of the following November, through her friend Amelot. Now talk of gratitude among relations, especially after all that I had done for them!

The rest of the letter relates to suggested changes in Montucla's history, then being revised by Lalande, and need not concern us. Enough has been given to allow us to picture to ourselves a very human man, enduring very human ills, and approaching his end with very human complaints,—a man the exact antithesis of what his contemporary conventional portraits show to the world at large.

QUESTIONS AND DISCUSSIONS

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

REPLIES.

36, 1. [*1919*, 69].[9] For what values of n can $\cos(2\pi/n)$ be expressed in the form $(a + \sqrt{b})/c$ where a, b and c are integers?

I. REPLY BY R. S. UNDERWOOD, Alabama Polytechnic Institute.

Using the formula[10]

$$2\cos n\theta = (2\cos\theta)^n - \frac{n}{1}(2\cos\theta)^{n-2} + \frac{n(n-3)}{2!}(2\cos\theta)^{n-4} - \frac{n(n-4)(n-5)}{3!}(2\cos\theta)^{n-6} + \cdots,$$

we get

$$2 = (2\cos 2m\pi/n)^n - \frac{n}{1}(2\cos 2m\pi/n)^{n-2} + \cdots;$$

[1] Jean Sylvain Bailly, born at Paris in 1736, executed at Paris in 1793. Arago's biography is well known. The misplaced pride consisted in standing for moderate principles against a bloodthirsty mob. Both he and Condorcet were prominent as mathematicians and as statesmen.

[2] Marie Jean Antoine Nicolas Caritat, Marquis de Condorcet, born at Ribemont in 1743; died at Bourg-la-Reine in 1794, a suicide by poison, in prison, to prevent being taken to Paris for execution.

Tallentyre wrote of him: "Since he never gave himself blindly to any one faction, all factions have distrusted and condemned him. To the Royalist he is a Revolutionist; to the Revolutionist he is an aristocrat. The thinker cannot forgive him that his thought led him to deeds and words; the man of action cannot forget that he was a thinker and dreamer to the end.

[3] Achille Pierre Dionis du Séjour (1734–1794), a celebrated astronomer. He fled to the country during the Terror, concealed himself, and died there.

[4] The recipient of the letter (apparently) has added the words, "He died of fear of death."

[5] The Marquis de Lesseville, now living at Châlons sur Marne, tells me that this was probably his great grandmother. The family is an old one.

[6] Femelles, female beasts, a term of great contempt.

[7] This shows that for the last twenty years of his life Montucla's family life was unpleasant. The event was only sixteen years after his marriage.

[8] Order for arrest.

[9] The other parts of this question were answered by the proposer, Professor Harris Hancock, *1919*, 292–295.

[10] See Problems—Notes 3, *1921*, 38.

so that the values of $2\cos(2m\pi/n)$ for $m = 1, 2, \cdots, n$ are the roots of the equation

$$\left\{ x^n - nx^{n-2} + \frac{n(n-3)}{2} x^{n-4} - \cdots \right\} - 2 = 0.$$

Since the coefficients in this equation are integers, the coefficients k and k_1 of any rational quadratic factor $x^2 + kx + k_1$ must also be integers. We may reject values of k and k_1 which make the roots of $x^2 + kx + k_1 = 0$ imaginary or greater than 2 in absolute value. Furthermore all the n roots of the original equation will be needed for the values of $2\cos(2m\pi/n)$ when $m = 1, 2, \cdots, n$; hence neither one of the roots of the quadratic can be greater than 2.

With this restriction, the possible irreducible quadratic factors are limited to[1] $x^2 - 2$, $x^2 - 3$, and $x^2 \pm x - 1$, while the only possible rational linear factors are x, $x \pm 1$, and $x \pm 2$. Hence the only possible rational and quadratic surd values for $\cos(2m\pi/n)$ are 0, $\pm 1/2$, ± 1, $\pm \sqrt{2}/2$, $\pm \sqrt{3}/2$, and $(\pm 1 \pm \sqrt{5})/4$; and the only values of n for which $\cos(2\pi/n)$ is of the form $(a + \sqrt{b})/c$, including the case $b = 0$, are 1, 2, 3, 4, 5, 6, 8, 10, and 12.

II. Remarks by the Editor.

It is evident that Professor Underwood's method may be used to find the positive integers n for which $\cos(2\pi/n)$ satisfies an equation of given degree k with integral coefficients. It is necessary that $2\cos(2\pi/n)$ satisfy an equation of the form

$$x^k + a_1x^{k-1} + a_2x^{k-2} + \cdots = 0, \tag{1}$$

in which (*a*) $a_1, a_2, \cdots$ are integers, and (*b*) all roots are real and not greater than 2 in absolute value. On account of (*b*), $|a_1| \leqq 2k$, $|a_2| \leqq 4\binom{k}{2}$, etc. Hence the number of equations (1) satisfying (*a*) and (*b*) is finite, and the possible equations may be examined in succession.

It will be noticed that in the cases $k = 1$ and $k = 2$ all "possible" equations led to actual solutions, and it might be thought that in the higher cases some difficulty would occur in recognizing those equations which, while conforming to (*a*) and (*b*), failed to satisfy the trigonometric test. Such difficulty however would not occur in view of the following interesting theorem, due to Kronecker:[2]

If an equation (1) *satisfies* (*a*) *and* (*b*), *each of its roots is of the form* $2\cos(2m\pi/n)$, *where* m *and* n *are integers.*

For proof, let $x = 2\cos\theta$, and let $y = 2\cos 2\theta = x^2 - 2$. On eliminating x between this equation and (1) by any of the ordinary methods, we get an equation in y of type (1) which is seen to satisfy (*b*). From the nature of the transformation it also satisfies (*a*). Hence, by applying successively transformations $z = y^2 - 2$, $u = z^2 - 2$, etc., we obtain always equations of type (1) satisfying (*a*) and (*b*). But such equations are finite in number. Therefore eventually we obtain two that are identical, and have the same roots. Now it may be that the roots of these two equations do not correspond to themselves, but are permuted by the transformations. In such a case we may apply more transformations until the permutation has been made $k!$ times, when every root will return to its own position. It will therefore happen that after, let us say, p and q transformations any root $2\cos\theta$ of (1) will be transformed into the same number. Hence $2^p\theta = 2r\pi \pm 2^q\theta$, where r is an integer and $p \neq q$; and therefore θ is commensurable with 2π.

The method used hitherto would prove very laborious for values of k exceeding 2 or 3; but the question may be settled in another way by means of the theory of the primitive nth roots of unity. If the prime divisors of n are $p, q, \cdots$, the number of primitive roots of $x^n = 1$ is $\varphi(n) = n(1 - 1/p)(1 - 1/q)\cdots$, being the number of positive integers prime to and not greater than n. They satisfy an equation $f(x) = 0$ of degree $\varphi(n)$, having integral coefficients. If n is greater than 2, $\varphi(n)$ is even, and the coefficients of $f(x)$ are unaltered by writing in reversed order; so that the transformation $y = x + x^{-1}$ leads to an equation in y of degree $\frac{1}{2}\varphi(n)$. But one of the roots in x is $e^{2i\pi/n}$; and therefore one of the roots in y is $2\cos(2\pi/n)$. It follows that $\cos(2\pi/n)$ is a root of a rational equation of degree $\frac{1}{2}\varphi(n)$.

[1] The question of rational linear factors (and therefore of rational cosines) was discussed by Professor Underwood in a recent issue of this Monthly (*1921*, 374).—Editor.

[2] "Zwei Sätze über Gleichungen mit ganzzahligen Coefficienten", *Journal für die reine und angewandte Mathematik*, vol. 53, p. 173; *Werke*, vol. 1, 1895, p. 107. Kronecker deduced the present theorem from another concerned with roots of unity; but his method is essentially similar to that of the direct proof given above.

The question now is whether $\cos(2\pi/n)$ can be a root of a rational equation of lower degree. Let it be a root of an equation of degree k. The substitution of the exponential form for the cosine then shows that $e^{2i\pi/n}$ is a root of a rational equation of degree $2k$. But the equation of primitive nth roots of unity is not rationally reducible;[1] that is, $e^{2i\pi/n}$ is not a root of any rational equation of degree less than $\varphi(n)$. Therefore k is not less than $\frac{1}{2}\varphi(n)$. We have then the theorem: *If n is an integer greater than 2, $\cos(2\pi/n)$ is a root of an irreducible equation of degree $\frac{1}{2}\varphi(n)$.*

The simpler cases are:

$\varphi(n) = 2$ when $n = 3, 4, 6$; $\cos(2\pi/n)$ rational, as also for $n = 1, 2$.
$\varphi(n) = 4$ when $n = 5, 8, 10, 12$; $\cos(2\pi/n)$ a quadratic surd.
$\varphi(n) = 6$ when $n = 7, 9, 14, 18$; $\cos(2\pi/n)$ a cubic surd.
$\varphi(n) = 8$ when $n = 15, 16, 20, 24, 30$; $\cos(2\pi/n)$ a quartic surd.

DISCUSSIONS.

Professor Hathaway obtains an integral reduction formula which includes as special cases the formulas usually given in works on the integral calculus.[2] In a note following the paper it is shown how the formula may be regarded as a transform of one of these special cases. This, of course, does not prevent it from being also a generalization.

A General Type of Reduction Formula.

By A. S. Hathaway, Rose Polytechnic Institute.

Let X, Y, Z be functions of a single variable, such that

$$AX^2 + BY^2 + CZ^2 = 0, \tag{1}$$

A, B, C being constants. By differentiation,

$$AXdX + BYdY + CZdZ = 0; \tag{2}$$

and from (1) and (2)

$$\frac{YdZ - ZdY}{AX} = \frac{ZdX - XdZ}{BY} = \frac{XdY - YdX}{CZ} = dT \text{ (for brevity).} \tag{3}$$

The general integral considered is

$$\beta(L, M, N) = \int X^L Y^M Z^N dT, \text{ where } L + M + N + 1 = 0. \tag{4}$$

The integral is homogeneous of order zero. It is not altered by substituting for X, Y, Z any variable common multiples of them, VX, VY, VZ. These also satisfy (1) and (3).

Further the integral is not altered by permuting X, Y, Z, concurrently with A, B, C and L, M, N; except that the sign is changed when the permutation is not cyclic (from the definition of dT). This means that any formula of expansion of the integral in powers of two bases is equivalent to six different formulas

[1] Dedekind's proof for the general case (n composite) is given in H. Weber, *Lehrbuch der Algebra*, volume 1 (Braunschweig, 1898), p. 596, or *Traité d'Algèbre Supérieure* (French translation by J. Griess, Paris, 1898), p. 636. For proof by Arndt, see P. Bachmann, *Die Lehre von der Kreistheilung* (Leipzig, 1872), p. 38.

[2] W. A. Granville, *Elements of the Differential and Integral Calculus*, Boston, 1911, pp. 350-360.